Extended resolvent approach to Inverse Scattering in multidimensions

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Building Inverse Scattering in two dimensions

Let us consider the KPI equation

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2}, \qquad x = (x_1, x_2),$$

a prototype of integrable equations in 2+1 dimensions.

It is associated to the Nonstationary Schrödinger operator

$$\mathcal{L}(x, i\partial_x) = i\partial_{x_2} + \partial_{x_1}^2 - u(x).$$

Being a generalization of the KdV equation, it admits solutions behaving at space infinity like the solutions of the KdV equation.

Any effort of building Inverse Scattering for solutions with constant behavior along some rays in the x-plane clashes with unsuccessful attempts to regularize the divergent integral equations defining the Jost solutions of $\mathcal{L}$.

One needs to proceed in two successive steps: first by considering the pure N soliton solution u_N and afterwards by adding an arbitrary smooth decaying background u' getting $u = u_N + u'$.

The nature of $\mathcal{L}$ is explored in the two cases by considering the entire family of its Green's functions in a very general class. In our language this corresponds to consider the extended resolvent of $\mathcal{L}$.

First we get the extended resolvent for the pure N soliton solution. Since its expression is totally explicit, when we introduce the perturbation we can deal successfully with the singularities due to the constant behavior at large space.

Extended operators and resolvent

For any differential operator $\mathcal{L}(x, i\partial_x)$ we introduce its **extension**

$$\mathbf{L}(x, x'; \mathbf{q}) \equiv \mathcal{L}(x, i\partial_x + \mathbf{q})\delta(x - x'), \qquad \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{C}^2.$$

By using the Fourier transform we can write

$$\mathbf{L}(x, x'; \mathbf{q}) = \frac{1}{(2\pi)^2} \int d\alpha\, e^{-i\alpha(x-x')} \mathcal{L}(x, \alpha + \mathbf{q}), \quad \alpha = (\alpha_1, \alpha_2).$$

In KPI case

$$\mathcal{L}(x, \mathbf{q}) = \mathbf{q}_2 - \mathbf{q}_1^2 - u(x).$$

By considering not just a polynomial $\mathcal{L}(x, \mathbf{q})$ in $\mathbf{q}$ but a tempered distribution $\mathcal{P}(x, \mathbf{q})$ in its six real variables we introduce more general operators, the **extended operators**

$$\mathbf{A}(x, x'; \mathbf{q}) = \frac{1}{(2\pi)^2} \int d\alpha\, e^{-i\alpha(x-x')} \mathcal{P}(x, \alpha + \mathbf{q})$$

Notice that

$$\mathbf{A}(x, x'; \mathbf{q}) = e^{i\mathbf{q}_{\Re}(x-x')} A(x, x'; q), \qquad q \equiv \mathbf{q}_{\Im}.$$

The extended operators generalize the pseudo-differential operators in two respects: they depend on a spectral parameter $\mathbf{q}$ and belong to the space of tempered distributions, which is larger than the functional space generally used.

$\mathcal{P}(x, \mathbf{q})$ generalizes what the mathematicians call the symbol of a pseudo-differential operator. This generalization is essential since just its dependence on the spectral parameter $\mathbf{q}$ allows us to get from some special symbols after the reduction

$$\mathbf{q} = \ell(\mathbf{k}) \equiv (\mathbf{k}, \mathbf{k}^2), \quad \mathbf{k} \in \mathbb{C}$$

the Jost solutions.

It is useful to consider also the Fourier transform of the symbol $\mathcal{P}(x, \mathbf{q})$

$$A(p; \mathbf{q}) = \frac{1}{(2\pi)^2} \int dx e^{ipx} \mathcal{P}(x, \mathbf{q}), \qquad p = (p_1, p_2).$$

Then, we can consider $A(x, x'; q)$ and $A(p; \mathbf{q})$ as two representations of the operator $A(q)$ in the x and in the p-space. The p-space is more suitable for studying analyticity properties, while boundedness is more easily studied in the x-space.

For generic operators $A(q)$ and $B(q)$ with kernels $A(x, x'; q)$ and $B(x, x'; q)$ we introduce the standard composition law

$$(AB)(x, x'; q) = \int dx''\, A(x, x''; q)\, B(x'', x'; q),$$

if the integral exists in terms of distributions.

The main object of our approach is the extended resolvent (or resolvent for short) $M(q)$ of the operator $L(q)$, which is defined as the inverse of the operator L, i.e.,

$$LM = ML = I, \quad I(x, x'; q) = \delta(x - x').$$

The Hilbert identity

$$M'(q) - M(q) = -M'(q)(L'(q) - L(q))M(q),$$

satisfied by two extended differential operators $L(q)$ and $L'(q)$ and their resolvent $M(q)$ and $M'(q)$ is the main instrument of our construction.

Hat operators and Green's functions

It is convenient to introduce the hat operators

$$\widehat{A}(x, x'; q) = e^{q(x-x')} A(x, x'; q).$$

The hat operator of a differential operator is given by

$$\widehat{L}(x, x'; q) = \mathcal{L}(x, \partial_x)\delta(x - x'),$$

i.e. it coincides with the initial operator $\mathcal{L}$ and thus it is independent of q. But the hat version of a generic operator depends on q.

By considering the hat version of the equation defining the resolvent we get

$$\mathcal{L}(x, \partial_x)\widehat{M}(x, x'; q) = \delta(x - x') = \mathcal{L}^{\mathrm{d}}(x', \partial_{x'})\widehat{M}(x, x'; q),$$

where $\mathcal{L}^{\mathrm{d}}(x', \partial_{x'})$ is the dual to $\mathcal{L}(x, \partial_x)$.

We use in the following the shortened notation

$$\overrightarrow{\mathcal{L}}\,\widehat{M}(q) = I = \widehat{M}(q)\overleftarrow{\mathcal{L}}\,.$$

$\widehat{M}(q)$ is a two-parameter set of Green's functions of $\mathcal{L}$.

Jost solutions and Spectral Data

We expect that reductions of the resolvent for which the symbol $\mathcal{L}_0(x, \mathbf{q}) = \mathbf{q}_2 - \mathbf{q}_1^2$ of the bare operator $\mathcal{L}_0 = i\partial_{x_2} + \partial_{x_1}^2$ is identically zero are of special interest. In fact the Green's functions obtained by considering the reductions

$$\mathcal{G}(\mathbf{k}) = \widehat{M}(q)\mid_{q=\ell_{\Im}(\mathbf{k})}, \quad \ell(\mathbf{k}) = (\mathbf{k}, \mathbf{k}^2), \quad \mathbf{k} \in \mathbb{C},$$

$$\mathcal{G}_\pm = \lim_{q_2 \to \pm 0} \lim_{q_1 \to 0} \widehat{M}(q).$$

furnish, respectively, the Jost and advanced/retarded solutions

$$\Phi(\mathbf{k}) = \mathcal{G}(\mathbf{k}) \overleftarrow{\mathcal{L}}_0 \Phi_0(\mathbf{k}), \qquad \Phi_\pm(k) = \mathcal{G}_\pm \overleftarrow{\mathcal{L}}_0 \Phi_0(k)$$

where

$$\mathcal{L}_0 = i\partial_{x_2} + \partial_{x_1}^2, \qquad \Phi_0(\mathbf{k}) = e^{-i\ell(\mathbf{k})x}.$$

For u **decaying at large** x, using the Hilbert identity, we can obtain the Green's function $\mathcal{G}(\mathbf{k})$ by dressing the Green's function $\mathcal{G}_0(\mathbf{k})$ of the bare operator $\mathcal{L}_0$

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_0(\mathbf{k}) + \mathcal{G}_0(\mathbf{k}) u\, \mathcal{G}(\mathbf{k})$$

where u is the multiplication operator with kernel $u(x, x'; q) = u'(x)\delta(x - x')$. Inserting into the definition of $\Phi(\mathbf{k})$ we get the usual integral equation defining it

$$\Phi(\mathbf{k}) = \Phi_0(\mathbf{k}) + \mathcal{G}_0(\mathbf{k}) u \Phi(\mathbf{k})$$

Analogously, for $\mathcal{G}_\pm$ and $\Phi_\pm(k)$.

The spectral data are introduced by relating Jost and advanced/retarded solutions

$$\Phi^{\sigma}(k) = \int dp\, \Phi_{\pm}(p)\mathcal{R}^{\sigma}_{\pm}(p,k), \quad \sigma = +,-, \qquad k,p \in \mathbb{R}, \sigma = +,-$$

where

$$\Phi^{\pm}(k) = \lim_{\mathbf{k}\to k \pm i0} \Phi(\mathbf{k})$$

Then, the discontinuity of the Jost solution across the real axis is given by

$$\Phi^{\sigma}(k) = \int dp\, \Phi^{-\sigma}(p)\mathcal{F}^{-\sigma}(p,k),$$

where

$$\mathcal{F}^{\sigma}(k,k') = \int dp\, \overline{\mathcal{R}^{-\sigma}_{\pm}(p,k)}\mathcal{R}^{-\sigma}_{\pm}(p,k').$$

Independence from $\pm$ furnishes the characterization equations for spectral data.

Expressions for spectral data are obtained by using the following formula obtained from the Hilbert identity

$$\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm} = \mathcal{G}_{\pm}\overleftarrow{\mathcal{L}}_0 \left(\mathcal{G}^{\sigma}_0(k) - \mathcal{G}_{0,\pm}\right) \overrightarrow{\mathcal{L}}_0\mathcal{G}^{\sigma}(k),$$

which relates $\mathcal{G}^{\sigma}(k) - \mathcal{G}_{\pm}$ to the difference $\mathcal{G}^{\sigma}_0(k) - \mathcal{G}_{0,\pm}$ of bare Green's functions.

If the potential is not decaying at large x all integral equations above are diverging. Let us consider the case $u = u_N + u'$ where u_N is the potential for N solitons and u' is an arbitrary "small" decaying function. Then, the resolvent $M(q)$ of $L(q)$ with potential $u = u_N + u'$ can be obtained by dressing the resolvent $M_N(q)$ of $L_N(q)$ with potential u_N

$$M(q) = M_N(q) + M_N(q)u'M(q)$$

which is a well defined integral equation since the bad behaviour of $M_N(q)$ is cured by the smoothness of u',

Then, by taking the reduction $q = (\mathbf{k}_\Im, 2\mathbf{k}_\Im\mathbf{k}_\Re)$ we get for the Green's function

$$\mathcal{G}(\mathbf{k}) = \mathcal{G}_N(\mathbf{k}) + \mathcal{G}_N(\mathbf{k})u'\mathcal{G}(\mathbf{k}),$$

The Jost solution can be written as

$$\Phi(\mathbf{k}) = \mathcal{G}(\mathbf{k})\overleftarrow{\mathcal{L}}_N\varphi(\mathbf{k}).$$

where $\varphi(\mathbf{k})$ is the Jost solution for N solitons. Inserting the integral equation for $\mathcal{G}(\mathbf{k})$ above we get

$$\Phi(\mathbf{k}) = \varphi(\mathbf{k}) + \mathcal{G}_N(\mathbf{k})u'\Phi(\mathbf{k})$$

which is a well defined integral equation.

Analogously for the advanced/retarded solutions,

Resolvent for N solitons

By using recursively a binary Darboux transformation one can construct explicitly u_N and its Jost solutions.

However, this recursive procedure seems not to be generalizable to the construction of the extended resolvent.

Therefore, we bypass the recursive procedure and build up directly u_N and Jost solutions by using a so called twisting operator ζ which is isometric but not unitary

$$\zeta^\dagger \zeta = I, \qquad \zeta \zeta^\dagger \neq I$$

and transforms the **extended** nonstationary Schrödinger bare operator L_0 to L_N with potential u_N according to the formula

$$L_N \zeta = \zeta L_0.$$

Once obtained ζ the resolvent is given by

$$M_N \zeta = \zeta M_0.$$

I skip heavy procedure needed for getting ζ and, then, M_N.

One gets M_N as a bilinear form in terms of the Jost solutions.

$$\widehat{M}_N(x,x';q) = \frac{\operatorname{sgn}(x_2 - x_2')}{2\pi i}\Bigl[\int d\alpha\, \theta\bigl((q_2 - 2\alpha q_1)(x_2 - x_2')\bigr) \times$$

$$\times\, t(\alpha + iq_1)\varphi(x;\alpha + iq_1)\psi(x';\alpha + iq_1) +$$

$$+ \sum_{n=1}^{N} \vartheta_n(q_1)\theta((q_2 - 2\lambda_{n\Re} q_1)(x_2 - x_2'))\varphi_n(x,q_1)\psi_n(x',q_1)\Bigr]$$

where λ_n ($\lambda_{n\Im} > 0$) are the discrete spectral data, $\varphi(x,\mathbf{k})$ is the Jost solution,

$$\varphi_n(x,q_1) = \theta(\lambda_{n\Im} - |q_1|)\Bigl\{\theta(q_1\lambda_{n\Im})\varphi(x,\lambda_n) + \theta(-q_1\lambda_{n\Im})\varphi(x,\overline{\lambda}_n)\Bigr\}$$

are additional Jost solutions that we call auxiliary, $t(\mathbf{k})$ is the transmission coefficient

$$t(\mathbf{k}) = \prod_{n=1}^{N}\left(\frac{\mathbf{k} - \overline{\lambda}_n}{\mathbf{k} - \lambda_n}\right)^{\operatorname{sgn}(\mathbf{k}_\Im)}$$

and

$$\vartheta_n(q_1) = -i\operatorname{sgn} q_1\theta(q_1)t_n + \theta(-q_1)\bar{t}_n, \quad t_n = \operatorname*{res}_{\mathbf{k}=\lambda_m} t(\mathbf{k}).$$

The normalization coefficients are defined by

$$\varphi(x,\lambda_n) = i\sum_{m=1}^{N}\varphi(x,\overline{\lambda}_m)\bar{t}_m C_{mn}.$$

ψ and ψ_n are the corresponding dual Jost solutions.

The Green's function $G_N(\mathbf{k})$ and consequently the Jost solution $\Phi(\mathbf{k})$ is analytical in the complex $\mathbf{k}$-plane with a discontinuity across the real axis and the segments $\mathbf{k} = \boldsymbol{\lambda}_{n\Re} + i\,\mathbf{k}_{\Im}$, $|\,\mathbf{k}_{\Im}\,| \leq \lambda_{n\Im}$, with a log singularity at the end points. Spectral data relate the limiting values of the Jost solution at the two sides of these cuts.

The study of singularities, the definition of perturbed advanced/-retarded $\Phi_{\pm}$ and auxiliary Jost solutions Φ_n, the derivation of spectral data, their characterization equations, their time evolution, the solution of the inverse problem, all this is a long story.

People interested can look at the paper.

The same procedure can be extended to KPII.