Dispersive shock waves: an analytic approach through the Inverse Scattering Transform

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A Bose-Einstein condensate (BEC) is a quantum fluid that gives rise to interesting shock-wave nonlinear dynamics. Recent experiments depict a BEC that exhibits behavior similar to that of a shock wave in a compressible gas, e.g., traveling fronts with steep gradients. However, the governing Gross-Pitaevskii (GP) equation that describes the mean field of a BEC admits no dissipation, hence classical dissipative shock solutions do not explain the phenomena. Instead, wave dynamics with small dispersion is considered and it is shown that this provides a mechanism for the generation of a dispersive shock wave (DSW).
The physical motivation

**Classic Shock Waves vs Dispersive Shock Waves**

**CSWs**

A (classic) shock wave in a compressible fluid is characterized by a steep jump in gas velocity, density, and temperature across which there is a dissipation of energy due fundamentally to collisions of particles.

**DSWs**

A different type of shock wave develops when the shock front is dominated not by dissipation but rather dispersion, e.g. in a quantum fluid that is a Bose-Einstein condensate (BEC), or in the propagation of light through a nonlinear, defocusing medium.

For example, one can observe shock-induced dynamics in BEC when a slow-light technique is used to produce a sharp density depression in the condensate.
A DSW has two associated speeds (front and rear of the oscillatory region) and a large amplitude oscillatory structure. Furthermore, the oscillatory region expands linearly in size as time increases.

On the contrary, a dissipative shock propagates with constant speed but the front remains localized in space.

Locally, these dispersive shock waves (DSWs) with large amplitude oscillations and two associated speeds bear little resemblance to their classical dissipative counterparts. However, a comparison is possible when one considers a mean-field theory, corresponding to the average of a DSW. For instance, although interactions of DSWs can be quite complicated, asymptotically for large time they are analogous to the interaction of two classical, viscous shock waves (VSWs) with an interaction region described by modulated quasi-periodic waves.
The model equation

The model equation is the 1D defocusing NLS equation (also known as the Gross-Pitaevskii (GP) equation)

\[ i\epsilon \psi_t = -\epsilon \psi_{xx} + |\psi|^2 \psi \]

for the mean field of a quasi-1D BEC, or the slowly varying envelope of the electromagnetic field propagating through a Kerr medium (with \( t \) replaced by the propagation distance).

The small parameter \( \epsilon \) is inversely proportional to the number of atoms in the BEC or, after rescaling, inversely proportional to the maximum initial intensity of the electromagnetic field.

The NLS equation can also be represented in a form analogous to the Euler equations of gas dynamics.
Inserting the transformation

\[ \Psi(x, t) = \sqrt{\rho(x, t)} \exp \left[ i \int_0^x u(x', t) \, dx'/\epsilon \right] \]

into the first two local conservation laws for NLS yields

\[ \rho_t + (\rho u)_x = 0 \]

\[ (\rho u)_t + \left( \rho u^2 + \frac{1}{2} \rho^2 \right)_x = \frac{\epsilon^2}{4} (\rho \log \rho)_{xx} \]

analogous to the Navier-Stokes equations for a perfect gas with fluid density (height) \( \rho \) and local fluid velocity \( u \), except that the viscous term has been replaced by the dispersive term with coefficient \( \epsilon^2/4 \).

For small dispersion, one expects the generation of small wavelength \( O(\epsilon) \) oscillations near a steep gradient in the fluid variables.
We consider, as an example, the problem of a “piston” moving with constant speed into a steady, dispersive fluid: e. g. a Bose-Einstein condensate or light propagating through a nonlinear, defocusing medium. The piston in this case is a step potential that moves with uniform velocity. This potential could be realized in a BEC with a repulsive dipole beam and in nonlinear optics with a local change in the index of refraction. One expects, in analogy with the classical, viscous case, the generation of a dispersive shock wave.

We assume the piston strength is large for $x < v_p t$, $v_p$ being the piston velocity, so there is negligible density there. Then we assume there is a jump from zero density to the nonzero value $\rho_L$ with a fluid velocity $u_L$. 
In practise we are assuming a step initial condition. At the time $t = 0$ we assume that there is a discontinuity in the fluid variables given by:

$$
\rho(x, 0) = \begin{cases} 
\rho_L & x = 0 \\
\rho_R & x < 0
\end{cases}
\quad u(x, 0) = \begin{cases} 
v_L & x = 0 \\
0 & x < 0
\end{cases}
$$

At time $t > 0$ a dispersive shock wave appears, characterized by expanding, modulated, periodic 1-phase waves.
Asymptotic solutions for this problem have been calculated using Whitham averaging theory for a “piston” (step potential) moving with uniform speed into a dispersive fluid at rest. And it has been showed that these asymptotic results agree quantitatively with numerical simulations.

Here we aim at solving an initial value problem for the defocusing NLS equation with a “jump” initial condition using the Inverse Scattering Transform (IST), and in this way characterizing analytically the long-time asymptotics of two (or more) colliding plane waves as a function of the wave amplitudes and relative velocities.
Inverse Scattering Transform

A number of nonlinear evolution equations are “linearized” via the IST, i.e. associated with a pair of linear problems (Lax pair), a linear eigenvalue problem and an auxiliary problem, such that the given equation results as the compatibility condition between them. The nonlinear evolution equation is then called integrable. We say that the operator pair $X, T$ is a Lax pair for the nonlinear equation

$$q_t = F[x, t, q, q_x, q_{xx}, \ldots] \quad q = q(x, t),$$

if

$$v_x = Xv, \quad v_t = Tv$$

[X, T are in general matrix functions of $q, q_x, q_{xx}, \ldots$] and the compatibility [i.e., the equality of the mixed derivatives $v_{xt} = v_{tx}$] is identically satisfied provided $q$ solves the nonlinear PDE.
The solution of the Cauchy problem by IST proceeds in three steps, as follows:

1. the direct problem - the transformation of the initial data from the original “physical” variables \((q(x, 0))\) to the transformed “scattering” variables \((S(k, 0))\);

2. time dependence - the evolution of the transformed data often according to simple, explicitly solvable evolution equations (i.e., finding \(S(k, t))\);

3. the inverse problem - the recovery of the evolved solution \((q(x, t))\) from the evolved solution in the transformed variables \((S(k, t))\).

Both the direct and the inverse problem make use of the first operator in the Lax pair, so-called scattering problem. The time evolution is determined by the second operator in the Lax pair.
Scheme of the Inverse Scattering Transform: the “nonlinear” analog of a Fourier Transform
Let us formulate the IST for the defocusing NLS equation

\[ iq_t = q_{xx} - 2 |q|^2 q \]

with initial condition

\[ q(x, 0) = \begin{cases} 
A_1 e^{i\mu_1 x} & x < 0 \\
A_2 e^{-i\mu_2 x} & x > 0 
\end{cases} \]

representing two plane waves of different amplitudes and velocities colliding at \( t = 0 \).
The Lax pair for the de-focusing NLS is given by

\[ v_x = \begin{pmatrix} -ik & q \\ q^* & ik \end{pmatrix} v \]
\[ v_t = \begin{pmatrix} 2ik^2 + i |q|^2 & -2kq - iq_x \\ -2kq^* + iq_x^* & -2ik^2 - i |q|^2 \end{pmatrix} v. \]

For any time \( t \) and large \( x \), \( q(x,t) \) behaves asymptotically like a plane wave, i.e.

\[ q(x,t) \sim A_1 e^{i\mu_1 x - i\omega_1 t} \quad x \to -\infty \]
\[ q(x,t) \sim A_2 e^{-i\mu_2 - i\omega_2 t} \quad x \to +\infty \]

and using the Galilean invariance of NLS, we can always choose \( \mu_1 \) and \( \mu_2 \) such that for the dispersion relations \( \omega_1 = \omega_2 \equiv \omega \).
Let us introduce the following transformation

\[
\begin{align*}
\mathbf{v}(x, t) = & \begin{pmatrix}
e^{i(\mu_1 x - \omega t)/2} & 0 \\
0 & e^{-i(\mu_1 x - \omega t)/2}
\end{pmatrix} \mathbf{w}(x, t) & x < 0 \\
\mathbf{v}(x, t) = & \begin{pmatrix}
e^{i(-\mu_2 x - \omega t)/2} & 0 \\
0 & e^{i(-\mu_2 x - \omega t)/2}
\end{pmatrix} \mathbf{w}(x, t) & x > 0
\end{align*}
\]

[Note that the transformation is continuous at \( x = 0 \) for any \( t \)]

such that at \( t = 0 \) the scattering problem simplifies to

\[
\begin{align*}
\mathbf{w}_x = & \begin{pmatrix}
-ik + i\mu_2/2 & A_2 \\
A_2 & ik - i\mu_2/2
\end{pmatrix} \mathbf{w} & x > 0 \\
\mathbf{w}_x = & \begin{pmatrix}
-ik - i\mu_1/2 & A_1 \\
A_1 & ik + i\mu_1/2
\end{pmatrix} \mathbf{w} & x < 0
\end{align*}
\]

\( w \) is continuous at \( x = 0 \), however \( w_x \) is not

\[
\mathbf{w}_x \big|_{x=0^+} - \mathbf{w}_x \big|_{x=0^-} = \begin{pmatrix}
i\mu/2 & A_2 - A_1 \\
A_2 - A_1 & -i\mu/2
\end{pmatrix} \mathbf{w} \big|_{x=0}
\]

where \( \mu = \mu_1 + \mu_2 \).
If we look for solutions of the scattering problem in the form of plane waves, $e^{i\lambda x}$, we find that the eigenvalues are $\lambda = \pm \lambda_R$ for $x > 0$ and $\lambda = \pm \lambda_L$ for $x < 0$, with

$$\lambda_R^2 = (k - \mu_2/2)^2 - A_2^2$$
$$\lambda_L^2 = (k + \mu_1/2)^2 - A_1^2$$

$\lambda_R$ has two branch points at $k_R^\pm = \mu_2/2 \pm A_2$ and $\lambda_L$ has two branch points at $k_L^\pm = -\mu_1/2 \pm A_1$.

Depending on the values of $\mu/2 \equiv (\mu_1 + \mu_2)/2$ with respect to $A_1, A_2$ the two branches can be separate, overlapping partially or completely.
**Figure:** The branches for $\lambda_L$ and $\lambda_R$ when $A_1 - A_2 < \mu/2 < A_1 + A_1$ (left), $\mu/2 > A_1 + A_2$ (right) and $0 < \mu/2 < A_1 - A_2$ (bottom).
0 < A_1 - A_2 < - (\mu_2 + \mu_1)/2 < A_1 + A_2

Figure: The branches for \lambda_L and \lambda_R when A_1 - A_2 < -\mu/2 < A_1 + A_1 (left), -\mu/2 > A_1 + A_2 (right) and 0 < -\mu/2 < A_1 - A_2 (bottom).
So for $x > 0$ we can define two eigenfunctions

$$
\psi(x, k) = e^{i\lambda_R x} \begin{pmatrix} -iA_2 \\ \lambda_R + k - \mu_2/2 \end{pmatrix}, \quad \bar{\psi}(x, k) = e^{-i\lambda_R x} \begin{pmatrix} \lambda_R + k - \mu_2/2 \\ iA_2 \end{pmatrix}
$$

and similarly for $x < 0$

$$
\phi(x, k) = e^{-i\lambda_L x} \begin{pmatrix} \lambda_L + k + \mu_1/2 \\ iA_1 \end{pmatrix}, \quad \bar{\phi}(x, k) = e^{i\lambda_L x} \begin{pmatrix} -iA_1 \\ \lambda_L + k + \mu_1/2 \end{pmatrix}
$$

where the constant vectors are eigenvectors of the scattering problem in the corresponding half-line.
By Abel’s theorem, the eigenfunctions $\psi, \bar{\psi}$ are linearly independent for all $x \in \mathbb{R}$, and so are $\phi, \bar{\phi}$.

If we introduce scattering coefficients $a(k)$ and $b(k)$ such that

$$
\phi(x, k) = a(k)\bar{\psi}(x, k) + b(k)\psi(x, k)
$$

$$
\bar{\phi}(x, k) = \bar{a}(k)\psi(x, k) + \bar{b}(k)\bar{\psi}(x, k)
$$

imposing continuity of the eigenfunctions at $x = 0$ and the given "jump" for their derivatives at $x = 0$, we can obtain the explicit expression of the scattering coefficients at $t = 0$

$$
a(k) = \frac{A_2(\lambda_L + k + \mu_1/2) + A_1(\lambda_R - k + \mu_2/2)}{2A_2\lambda_R}
$$

$$
b(k) = i\frac{(\lambda_L + k + \mu_1/2)(\lambda_R - k + \mu_2/2) + A_1 A_2}{2A_2\lambda_R}
$$

The expressions for $\bar{a}(k)$ and $\bar{b}(k)$ are easily obtained taking into account that due to the symmetries in the scattering problem for real $k$, $\bar{a}(k) = a(k)$ and $\bar{b}(k) = b^*(k)$. 
The reflection coefficients

\[ \rho(k) = \frac{b(k)}{a(k)}, \quad \bar{\rho}(k) = \frac{\bar{b}(k)}{\bar{a}(k)}, \]

are then given by

\[ \rho(k) = i \frac{(\lambda_L + k + \mu_1/2)(\lambda_R - k + \mu_2/2) + A_1 A_2}{A_2(\lambda_L + k + \mu_1/2) + A_1(\lambda_R - k + \mu_2/2)} \]

\[ \bar{\rho}(k) = -i \frac{(\lambda_L + k + \mu_1/2)(\lambda_R - k + \mu_2/2) + A_1 A_2}{A_2(\lambda_L + k + \mu_1/2) + A_1(\lambda_R - k + \mu_2/2)} \]

and they are discontinuous across both cuts \( \lambda_L \) and \( \lambda_R \).

One can prove that the coefficient \( a(k) \) [and therefore \( \bar{a}(k) \)] never vanishes, and therefore with the prescribed initial conditions the scattering problem does not exhibit a discrete spectrum.
Behavior at the branch points

The behavior of the scattering coefficients at the branch points

\[
a(k)|_{k=\mu_2/2\pm A_2} = \frac{a_{\pm}}{\lambda_R} + O(1)
\]

\[
b(k)|_{k=\mu_2/2\pm A_2} = \frac{b_{\pm}}{\lambda_R} + O(1)
\]

with nonzero \(a_{\pm}, b_{\pm}\), in the generic case. One can check that \(a_{\pm} = \mp ia_{\pm}\) and as a consequence, the reflection coefficients are finite at the branch points of \(\lambda_R\), with

\[
\lim_{k \to \mu_2/2\pm A_2} \rho(k) = \mp i, \quad \lim_{k \to \mu_2/2\pm A_2} \bar{\rho}(k) = \pm i.
\]

Note that the coefficients \(a_{\pm}\) and \(b_{\pm}\) vanish iff \(A_1 - A_2 = \pm \mu/2\), in which case \(a(k)\) and \(b(k)\) are non singular near the corresponding branch points \(\mu_2/2 \pm A_2\), which means that \(k = \mu_2/2 + A_2\) or \(k = \mu_2/2 - A_2\) are virtual levels.
The equations defining the scattering coefficients also provide the starting point for the formulation of the inverse problem. The relation among the eigenfunctions can be formulated as a Riemann-Hilbert problem, which can be written in matrix form as

$$M_+ = M_- V_j \quad j = 0, \ldots, 3$$

where $j$ identifies a portion of the real axis, and for $x > 0$

$$M_+ = \begin{bmatrix} e^{i\lambda_R x} \phi/a & e^{-i\lambda_R x} \psi \end{bmatrix} \quad M_- = \begin{bmatrix} e^{i\lambda_R x} \bar{\psi} & e^{-i\lambda_R x} \bar{\phi}/\bar{a} \end{bmatrix}$$

A similar expression holds for $x < 0$. $M_+$ is an analytic function of $k$ on the upper half-plane and $M_-$ is analytic on the lower half-plane. Across the real axis, but off the branch cuts the jump matrix is given by

$$V_0 = \begin{pmatrix} 1 - \bar{\rho} \rho & -\bar{\rho} e^{-2i\lambda_R x} \\ \rho e^{2i\lambda_R x} & 1 \end{pmatrix}$$
Across each cut ($j = 1, 2$) we can write

\[
\begin{align*}
M_+ &= M_- V_1 \quad \text{on } \lambda_L \\
M_+ &= M_- V_2 \quad \text{on } \lambda_R
\end{align*}
\]

where $\pm$ denote the non-tangential limits from the left and right of the contours, and $V_1$ and $V_2$ are given by

\[
V_1 = \begin{bmatrix} 1 - \rho_+ \bar{\rho}_- & -\bar{\rho}_- e^{-2i\lambda_R x} \\ \rho_+ e^{2i\lambda_R x} & 1 \end{bmatrix}, \quad x > 0
\]

\[
V_2 = \frac{\lambda_R + k - \mu_2/2}{iA_2} \begin{bmatrix} (\rho_+ + \bar{\rho}_-) e^{2i\lambda_R x} & 1 \\ -1 & 0 \end{bmatrix}, \quad x > 0.
\]
Finally, when both $\lambda_L$ and $\lambda_R$ are discontinuous, the jump matrix is the same as the one across $\lambda_R$, i.e.

$$V_3 = \frac{\lambda_R + k - \mu_2/2}{iA_2} \begin{bmatrix} (\rho_+ + \bar{\rho}_-)e^{2i\lambda_R x} & 1 \\ -1 & 0 \end{bmatrix}, \quad x > 0,$$

except that in this case $\lambda_L$ also changes sign when considering $\bar{\rho}_-$, i.e. $\bar{\rho}_- = \bar{\rho}(k, -\lambda_R, -\lambda_L)$.

It is obvious that the structure of the RHP depends on the structure of the cuts, and therefore ultimately on the velocities and amplitudes of the interacting waves at time $t = 0$. 
Figure: The jump matrices when $A_1 - A_2 < \mu/2 < A_1 + A_2$ (left), $\mu/2 > A_1 + A_2$ (right) and $0 < \mu/2 < A_1 - A_2$ (bottom).
Figure: The jump matrices when $A_1 - A_2 < -\mu/2 < A_1 + A_2$ (left), $-\mu/2 > A_1 + A_2$ (right) and $0 < -\mu/2 < A_1 - A_2$ (bottom).
Using the second operator in the Lax pair, the time evolution of the scattering coefficients can be easily computed [first order ODE’s]

\[
\begin{align*}
    a(k, t) &= a(k, 0) e^{-2i[(k - \mu_1/2)\lambda_L - (k + \mu_2/2)\lambda_R]t} \\
    b(k, t) &= b(k, 0) e^{-2i[(k - \mu_1/2)\lambda_L + (k + \mu_2/2)\lambda_R]t}
\end{align*}
\]

[Note that both \(a(k)\) and \(b(k)\) depend explicitly on time] and

\[
\begin{align*}
    \rho(k, t) &= \rho(k, 0) e^{-4i(k + \mu_2/2)\lambda_R t} \\
    \bar{\rho}(k, t) &= \bar{\rho}(k, 0) e^{4i(k + \mu_2/2)\lambda_R t}
\end{align*}
\]

The potential is then reconstructed from the large \(k\)-asymptotics of the eigenfunctions. In particular,

\[
q(x, t) = -2 (M_1(x, t))_{12} e^{-i[-\mu_2x - \omega t]} \quad x > 0
\]

where \((M_1)_{12}\) is the 12-entry of \(M_1\), and \(M_1\) is the \(O(1/k)\) coefficient of the solution \(M\) of the RHP for \(x > 0\).
Summarizing, for $x > 0$ we have the following RH problem:

\[
V_0 = \begin{bmatrix}
1 - \rho \bar{\rho} & -\bar{\rho} e^{-2i f_R(k, \xi)t} \\
\rho e^{2i f_R(k, \xi)t} & 1
\end{bmatrix}
\] across the real axis off all cuts

\[
V_1 = \begin{bmatrix}
1 - \rho_+ \bar{\rho}_- & -\bar{\rho}_- e^{-2i f_R(k, \xi)t} \\
\rho_+ e^{2i f_R(k, \xi)t} & 1
\end{bmatrix}
\] across $\lambda_L$

\[
V_2 = \frac{\lambda_R + k - \mu_2/2}{i A_2} \begin{bmatrix}
(\rho_+ + \bar{\rho}_-) e^{2i f^+_R(k, \xi)t} & 1 \\
-1 & 0
\end{bmatrix}
\] across $\lambda_R$

\[
V_3 = \frac{\lambda_R + k - \mu_2/2}{i A_2} \begin{bmatrix}
(\rho_+ + \bar{\rho}_-) e^{2i f^+_R(k, \xi)t} & 1 \\
-1 & 0
\end{bmatrix}
\] across $\lambda_L \cap \lambda_R$

where $\xi = x/t$ (and $\xi > 0$ because $t > 0$), and

\[
f_R(k, \xi) = \lambda_R (\xi - 2k - \mu_2) \equiv (\xi - 2k - \mu_2) \sqrt{(k - \mu_2/2)^2 - A_2^2}.
\]

Similar expressions for $x < 0$, with $\lambda_R \to \lambda_L$ and $\mu_2 \to -\mu_1$. 
In order to study the decay and growth in time of the jump matrices, we have to analyze the sign of the functions $\text{Im} \ f_R(k, \xi)$, i.e.

$$f_R(k, \xi) = \lambda_R(\xi - 2k - \mu_2) \equiv (\xi - 2k - \mu_2)\sqrt{(k - \mu_2/2)^2 - A_2^2}$$

for $\xi > 0$.

For $k = \eta + i\nu$ with $|\nu| \ll 1$ (i.e., $k$ right above or below the real axis), we can expand $f_R(k, \xi)$ in a Taylor series with respect to $\nu$ and study the sign of its imaginary part.

The sign structure of $\text{Im} \ f_R$ depends on the possible configurations of the branch cuts.
Figure: The sign structure of $\text{Im } f_R$ for $\mu/2 > A_1 + A_2$ when $\xi > 2A_2 + 2\mu_2$ (top left), $2\mu_2 - 2A_2 < \xi < 2A_2 + 2\mu_2$ (top right) and $0 < \xi < 2\mu_2 - 2A_2$ (bottom).
Long-time asymptotics

The sign structure of $\text{Im } f_R$ for $A_1 - A_2 < \mu < A_1 + A_2$ when $\xi > 2A_2 + 2\mu_2$ (top left), $0 < 2\mu_2 - 2A_2 < \xi < 2A_2 + 2\mu_2$ (top right) and $0 < \xi < 2\mu_2 - 2A_2$ (bottom).

**Figure:** The sign structure of $\text{Im } f_R$ for $A_1 - A_2 < \mu < A_1 + A_2$ when $\xi > 2A_2 + 2\mu_2$ (top left), $0 < 2\mu_2 - 2A_2 < \xi < 2A_2 + 2\mu_2$ (top right) and $0 < \xi < 2\mu_2 - 2A_2$ (bottom).
Let us start by examining the region $\xi > 2\mu_2 + 2A_2 > 0$. All jump matrices except the one across $\lambda_R$ decay to the identity as $t \to +\infty$, and therefore do not contribute to leading order. The jump matrix on $\lambda_R$ will produce the leading-order solution and the other contours provide higher-order corrections. Moreover, at leading order the jump matrix across $\lambda_R$ is constant and the RHP is given by

$$V_2 = \frac{\lambda_R + k - \mu_2/2}{iA_2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The corresponding RHP [model RHP for this parameter region] can be solved explicitly and the region $\xi > 2\mu_2 + 2A_2$ in the $(x,t)$-plane is identified as a 0-phase region.
A similar investigation for the other regions in the \((x, t)\)-plane gives the following pictorial representation for the region of parameters where the two cuts are disjoint [region 1]:
This is a work in progress. We still have to complete the characterization of all regions of parameters for the amplitudes and velocities of the interacting plane waves.

The numerical simulations suggests the following scenario:

In the future, we plan to use the IST approach to investigate interactions of dispersive shock waves corresponding to more general discontinuous initial conditions.